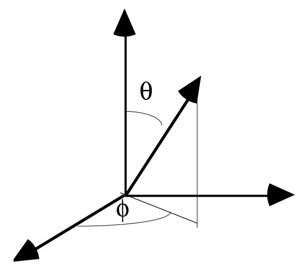
Physics 566: Quantum Optics I
Problem Set 1
Due Thursday, September 12, 2013

Problem 1: Some properties of spin 1/2 and the Bloch sphere (10 points)
Given a unit vector $\vec{\mathbf{e}}_n$, defined by angles θ and ϕ with respect to the polar axis z,



(a) Show that every pure state for a two-level system, $|\psi\rangle = \alpha|\uparrow_z\rangle + \beta|\downarrow_z\rangle$, is equivalent to a ket $|\uparrow_n\rangle$, defined as the spin-up state along an axis $\vec{\mathbf{e}}_n$. What are the angles θ and ϕ?

(b) Show that the one dimensional projector corresponding to measurement of $|\uparrow_n\rangle$ is,

$$|\uparrow_n\rangle\langle\uparrow_n| = \frac{1}{2}\left(\hat{1} + \vec{\mathbf{e}}_n \cdot \hat{\vec{\sigma}}\right).$$

(c) Show that the inner product between any two pure states is, $\left|\langle\uparrow_n|\uparrow_{n'}\rangle\right| = \cos(\Theta/2)$, where Θ is the angle between the directions $\vec{\mathbf{e}}_n$ and $\vec{\mathbf{e}}_{n'}$ in three dimensional space.

(d) Show that $\langle\uparrow_n|\hat{\vec{\sigma}}|\uparrow_n\rangle = \vec{\mathbf{e}}_n$, the Bloch vector.

Problem 2: Spin precession in a magnetic field - Heisenberg picture (10 Points)
Consider a spin 1/2 particle such as an electron in a magnetic field. Such a particle has an intrinsic magnetic moment, described by the operator, $\hat{\vec{\mu}} = -\gamma_s \hat{\mathbf{S}}$, where γ_S is known as the "gyromagnetic ratio", and $\mathbf{S} = \hat{S}_x \mathbf{e}_x + \hat{S}_y \mathbf{e}_y + \hat{S}_z \mathbf{e}_z$ in the spin 1/2 angular momentum operator. When placed in a magnetic field $\mathbf{B}$, the interaction energy is described by the Hamiltonian

$$\hat{H} = -\hat{\vec{\mu}} \cdot \mathbf{B}.$$

(a) Show that the Heisenberg equation of motion for the spin operator is

$$\frac{d\hat{\mathbf{S}}}{dt} = \vec{\Omega} \times \hat{\mathbf{S}}\text{, where } \vec{\Omega} = \gamma_s \mathbf{B}$$

Describe the physical meaning of this differential equation if we take **S** to be a classical angular momentum vector.

(b) Find the Heisenberg equations of motion for the spherical components $\hat{\sigma}_z, \hat{\sigma}_\pm$ (do this through direct commutation with the Hamiltonian and check with part (a)).

(c) Solve this equation for $\hat{\sigma}_x(t), \hat{\sigma}_y(t), \hat{\sigma}_z(t)$ in terms of the initial operators for the particular case the magnetic field is $\mathbf{B} = B_x \mathbf{e}_x + B_z \mathbf{e}_z$. Use this solution to find the trajectory of the Bloch vector $\mathbf{Q}(t) = \langle \vec{\sigma}(t) \rangle$ for the Heisenberg state $|\downarrow_z\rangle$ (this is the initial state in the Schrödinger picture). Sketch the trajectory on the Bloch sphere.

Problem 3: Qubits encoded in photon polarization (15 points)

The two orthogonal polarization states of a photon define a qubit. Let us define the standard basis

$$\frac{\mathbf{e}_H + i\mathbf{e}_V}{\sqrt{2}} : \text{right hand circular (positive helicity)} \Rightarrow |\uparrow_z\rangle$$
$$\frac{\mathbf{e}_H - i\mathbf{e}_V}{\sqrt{2}} : \text{left hand circular (negative helicity)} \Rightarrow |\downarrow_z\rangle$$

where $(\mathbf{e}_H, \mathbf{e}_V)$ are linear polarizations along some defined "horizontal" and "vertical" axes.

The Bloch sphere description of the polarization is known as the "Poincaré sphere", with each point on the surface representing a possible elliptical polarization. The three Cartesian coordinates of the Bloch vector are also known as the "Stokes parameters".

(a) To what polarization vectors do you associate $|\uparrow_x\rangle, |\downarrow_x\rangle$ and $|\uparrow_y\rangle, |\downarrow_y\rangle$? What is the nature of the polarization, linear, circular, elliptical, and along what direction if linear?

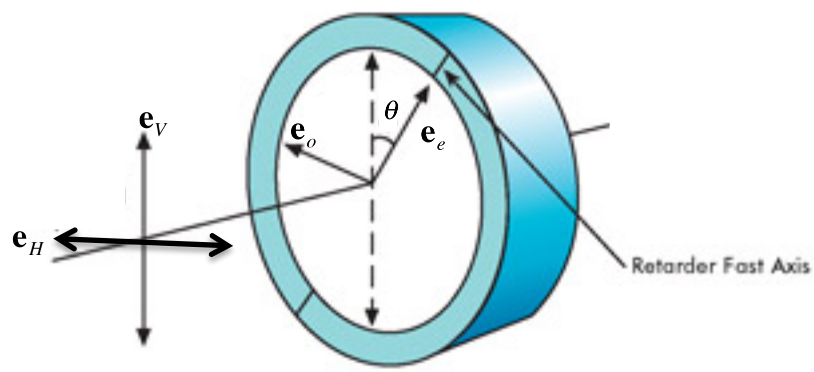
(b) Generalize: What is the polarization vector corresponding to an arbitrary state of the qubit, $|\uparrow_n\rangle$? Give the ellipticity and semi-major/minor axes of the ellipse in terms of the direction (θ,ϕ) of the state $|\uparrow_n\rangle$ on the Poincaré sphere?

(c) Sketch the Poincaré sphere, denoting the polarization states at the north and south pole, and at a few points along the equator as well as a few great circle a with constant latitude/longitude.

A wave plate is an optical element with birefringence, i.e., the index of refraction is in different along two orthogonal axes, “ordinary” and “extraordinary” m n_o, n_e. The result is that the phase shift imparted to the light depends on the polarization of the light with eigenvectors

$$\mathbf{e}_o \Rightarrow e^{i\frac{\omega}{c}n_o L}\mathbf{e}_o, \quad \mathbf{e}_e \Rightarrow e^{i\frac{\omega}{c}n_e L}\mathbf{e}_e$$

where L is the thickness of crystal. By orienting the crystal at an angle θ with respect to the H,V axes, one can transform the polarization state.



(d) Write the transformation of the polarization state by the wave plate as an SU(2) matrix acting on the Poincaré sphere. The matrix should be specified by two parameters, θ and $\Delta\phi = 2\pi\frac{(n_e - n_0)L}{\lambda}$, expressed in any representation you prefer.

(e) A quarter-wave plate has $L = \frac{\lambda}{4}(n_e - n_o)$; a half-wave plate has $L = \frac{\lambda}{2}(n_e - n_o)$.

How should the quarter-wave plate be oriented to transform horizontal polarization to circular polarization? What is the rotation on the Poincaré sphere?

How should a half-wave plate be oriented to transform horizontal polarization to vertical polarization? What is the rotation on the Poincaré sphere?

(f) Extra credit (5 points). Show that an arbitrary SU(2) transformation on the Poincaré sphere can be constructed using two quarter-wave plates and one half-wave plates.